Random Field Ising Model in four spatial dimensions and beyond

Víctor Martín-Mayor, Nikolaos G. Fytas (Coventry, UK), Nicolas Sourlas and Marco Picco (Paris)

Department of Theoretical Physics I, Complutense University, Madrid

Barcelona, September 23, 2015
Disorder may have dramatic effects in condensed matter physics. For instance, adding a small amount of Strontium of $La_2CuO_4$ turns an electrical insulator into a superconductor.
Disorder may have dramatic effects in condensed matter physics. For instance, adding a small amount of Strontium of $La_2CuO_4$ turns an electrical insulator into a superconductor.

Basic approach in fundamental Physics: identify minimal model for complex behavior (support: Universality and Wilson’s RG).
Disorder may have dramatic effects in condensed matter physics. For instance, adding a small amount of Strontium of $La_2CuO_4$ turns an electrical insulator into a superconductor.

Basic approach in fundamental Physics: identify minimal model for complex behavior (support: Universality and Wilson’s RG).

The Random-Field Ising Model (RFIM) is a cherished but still not completely understood model for the effects of disorder.
Introduction (II): The standard Ising model

\[ H = -J \sum_{x,y} s_x s_y - h \sum_x s_x , \quad s_x = \pm 1. \]
Introduction (II): The standard Ising model

\[ H = -J \sum_{x,y} s_x s_y - h \sum_x s_x, \quad s_x = \pm 1. \]
The Random-Field Ising model

\[ H = -J \sum_{x,y} s_x s_y - \sum_x h_x s_x , \quad s_x = \pm 1 . \]

Two conflicting terms in the Hamiltonian.
Introduction (III): The Random-Field Ising model

\[ H = -J \sum_{x,y} s_x s_y - \sum_x h_x s_x, \quad s_x = \pm 1. \]

- Two conflicting terms in the Hamiltonian.
- Found useful in a wild variety of contexts:
  - Antiferromagnets in an externally applied magnetic field.
  - Binary liquids in porous media
  - Colossal magnetoresistance oxides.
  - Ferroelectrics, . . .
In spite of innocent aspect, quintessential non-perturbative problem (e.g. the lower critical dimension paradox).
Introduction (III): The Random-Field Ising model

\[ H = -J \sum_{x,y} s_x s_y - \sum_x h_x s_x , \quad s_x = \pm 1. \]

- Two conflicting terms in the Hamiltonian.
- Found useful in a wild variety of contexts:
  - Antiferromagnets in an externally applied magnetic field.
  - Binary liquids in porous media
  - Colossal magnetoresistance oxides.
  - Ferroelectrics, ...
- In spite of innocent aspect, quintessential non-perturbative problem (e.g. the lower critical dimension paradox).
- Space dimension is an all-important variable.
The random-field Ising model (RFIM) is a long-standing problem of Statistical Mechanics (≈ 1700 papers in the years 1970 - 2012: source ISI WEB).
The random-field Ising model (RFIM) is a long-standing problem of Statistical Mechanics (≈ 1700 papers in the years 1970 - 2012: source ISI WEB).

Unusual RG: fixed point at zero temperature ($T = 0$) leading to hyper-scaling violations (exponent $\theta$).
Introduction (IV): some open problems

- The random-field Ising model (RFIM) is a long-standing problem of Statistical Mechanics (≈ 1700 papers in the years 1970 - 2012: source ISI WEB).
- Unusual RG: fixed point at zero temperature ($T = 0$) leading to hyper-scaling violations (exponent $\theta$).
- Some cherished concepts, i.e. the two-exponent scaling scenario ($\bar{\eta} = 2\eta$), have been recently questioned (Tissier and Tarjus, PRL 107, 041601 (2011)).
The random-field Ising model (RFIM) is a long-standing problem of Statistical Mechanics ($\approx 1700$ papers in the years 1970 - 2012: source ISI WEB).

Unusual RG: fixed point at zero temperature ($T = 0$) leading to hyper-scaling violations (exponent $\theta$).

Some cherished concepts, i.e. the two-exponent scaling scenario ($\bar{\eta} = 2\eta$), have been recently questioned (Tissier and Tarjus, PRL 107, 041601 (2011)).

Universality in terms of different random-field distributions has been severely questioned many times.
The random-field Ising model (RFIM) is a long-standing problem of Statistical Mechanics (∼ 1700 papers in the years 1970 - 2012: source ISI WEB).

Unusual RG: fixed point at zero temperature ($T = 0$) leading to hyper-scaling violations (exponent $\theta$).

Some cherished concepts, i.e. the two-exponent scaling scenario ($\tilde{\eta} = 2\eta$), have been recently questioned (Tissier and Tarjus, PRL 107, 041601 (2011)).

Universality in terms of different random-field distributions has been severely questioned many times.

We only have analytic control of the problem in very high space dimensions (upper critical dimension: $d = 6$).
The random-field Ising model (RFIM) is a long-standing problem of Statistical Mechanics (≈ 1700 papers in the years 1970 - 2012: source ISI WEB).

Unusual RG: fixed point at zero temperature ($T = 0$) leading to hyper-scaling violations (exponent $\theta$).

Some cherished concepts, i.e. the two-exponent scaling scenario ($\bar{\eta} = 2\eta$), have been recently questioned (Tissier and Tarjus, PRL 107, 041601 (2011)).

Universality in terms of different random-field distributions has been severely questioned many times.

We only have analytic control of the problem in very high space dimensions (upper critical dimension: $d = 6$). Understanding what happens upon varying $d$ is a critical issue.
Ingredients in our approach (I): The Hamiltonian and the $T = 0$ scenario.
Ingredients in our approach (I): The Hamiltonian and the $T = 0$ scenario.

$$\mathcal{H}^{(RFIM)} = -J \sum_{\langle x, y \rangle} S_x S_y - \sum_x h_x S_x; \; S_x = \pm 1; \; J > 0$$
Ingredients in our approach (I): The Hamiltonian and the $T = 0$ scenario.

\[
\mathcal{H}^{(\text{RFIM})} = -J \sum_{\langle x,y \rangle} S_x S_y - \sum_x h_x S_x, \quad S_x = \pm 1; \quad J > 0
\]

- Work at $T = 0$ using efficient optimization algorithms that calculate exact ground states (Middleton and Fisher, PRB 65, 134411 (2002)).
Ingredients in our approach (I): The Hamiltonian and the $T = 0$ scenario.

$$
\mathcal{H}^{(\text{RFIM})} = -J \sum_{\langle x, y \rangle} S_x S_y - \sum_x h_x S_x; \; S_x = \pm 1; \; J > 0
$$

- Work at $T = 0$ using efficient optimization algorithms that calculate exact ground states (Middleton and Fisher, PRB 65, 134411 (2002)).
Ingredients in our approach (II): Simulated (continuous) field distributions

\[
\begin{align*}
\text{double Gaussian (dG):} \\
W_{dG}(h_x; h_R, \sigma) &= \frac{1}{2} \sqrt{\frac{2}{\pi \sigma^2}} \left[ e^{-\frac{(h_x - h_R)^2}{2\sigma^2}} + e^{-\frac{(h_x + h_R)^2}{2\sigma^2}} \right] \\
\text{bimodal (b):} \\
\sigma &= 0 \\
\text{Gaussian (G):} \\
h_R &= 0 \\
dG(\sigma = 1) &\text{: bimodal - like continuous distribution} \\
\text{Poissonian (P):} \\
W_{P}(h_x; h_R) &= \frac{1}{2| h_R |} e^{-\frac{| h_x |}{h_R}} \\
\end{align*}
\]
Ingredients in our approach (II): Simulated (continuous) field distributions

- double Gaussian (dG):

\[
\mathcal{W}^{(dG)}(h_x; h_R, \sigma) = \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} \left[ e^{-\frac{(h_x-h_R)^2}{2\sigma^2}} + e^{-\frac{(h_x+h_R)^2}{2\sigma^2}} \right]
\]
Ingredients in our approach (II): Simulated (continuous) field distributions

- double Gaussian (dG):
  \[ \mathcal{W}^{(\text{dG})}(h_x; h_R, \sigma) = \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma^2} \left[ e^{-\frac{(h_x-h_R)^2}{2\sigma^2}} + e^{-\frac{(h_x+h_R)^2}{2\sigma^2}} \right] \]

- bimodal (b): \( \sigma = 0 \)
Ingredients in our approach (II): Simulated (continuous) field distributions

- double Gaussian (dG):
  \[
  W^{(dG)}(h_x; h_R, \sigma) = \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma^2} \left[ e^{-\frac{(h_x-h_R)^2}{2\sigma^2}} + e^{-\frac{(h_x+h_R)^2}{2\sigma^2}} \right]
  \]
  - bimodal (b): \(\sigma = 0\)
  - Gaussian (G): \(h_R = 0\)

V, M.-M., N.G.F., N.S., M.P.  RES Users' 2015: RFIM 4D and beyond
Ingredients in our approach (II): Simulated (continuous) field distributions

- double Gaussian (dG):

\[ \mathcal{W}^{(dG)}(h_x; h_R, \sigma) = \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} \left[ e^{-\frac{(h_x-h_R)^2}{2\sigma^2}} + e^{-\frac{(h_x+h_R)^2}{2\sigma^2}} \right] \]

- bimodal (b): \( \sigma = 0 \)
- Gaussian (G): \( h_R = 0 \)
- \( dG^{(\sigma=1)} \): bimodal - like continuous distribution

![Graph showing double Gaussian distribution](graph.png)
Ingredients in our approach (II): Simulated (continuous) field distributions

- double Gaussian (dG):
  \[ W^{(dG)}(h_x; h_R, \sigma) = \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} \left[ e^{-\frac{(h_x-h_R)^2}{2\sigma^2}} + e^{-\frac{(h_x+h_R)^2}{2\sigma^2}} \right] \]
  - bimodal (b): \( \sigma = 0 \)
  - Gaussian (G): \( h_R = 0 \)
  - \( dG^{(\sigma=1)} \): bimodal - like continuous distribution

- \( dG^{(\sigma=2)} \)

V,M.-M., N.G.F., N.S., M.P.
RES Users' 2015: RFIM 4D and beyond
Ingredients in our approach (II): Simulated (continuous) field distributions

- double Gaussian (dG):
  \[ \mathcal{W}^{(dG)}(h_x; h_R, \sigma) = \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} \left[ e^{-\frac{(h_x-h_R)^2}{2\sigma^2}} + e^{-\frac{(h_x+h_R)^2}{2\sigma^2}} \right] \]

- bimodal (b): \( \sigma = 0 \)
- Gaussian (G): \( h_R = 0 \)
- \( dG^{(\sigma=1)} \): bimodal - like continuous distribution

- \( dG^{(\sigma=2)} \)
- Poissonian (P):
  \[ \mathcal{W}^{(P)}(h_x; h_R) = \frac{1}{2|h_R|} e^{-|h_x|/h_R} \]
Ingredients in our approach (III): Computational scheme

Use fluctuation-dissipation formalism to:

- Compute connected correlation functions $\Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y}$.
- Compute as well disconnected correlations $G_{xy} = \langle S_x S_y \rangle$.

For either correlator $\to$ second-moment correlation length.

Perform re-weighting extrapolation on $h_R$.

Compute derivatives with respect to $h_R$ $\to$ estimation of the critical exponent $\nu$.

Obtain size-dependent effective exponents $\to$ control scaling corrections (make use of the quotient method).

Perform high statistics in both directions: $L \leq 192$ in $d = 3$ ($L \geq 60$ in $d = 4$), # disorder samples in the range $(10^{-50}) \times 10^6$. 

V, M.-M., N.G.F., N.S., M.P. 
RES Users' 2015: RFIM 4D and beyond
Use fluctuation-dissipation formalism to:

- Compute connected correlation functions $\Gamma_{xy} = \partial \langle S_x \rangle / \partial h_y$.
- Compute as well disconnected correlations $G_{xy} = \langle S_x S_y \rangle$.

For either correlator $\rightarrow$ second-moment correlation length.

Perform re-weighting extrapolation on $h_R$.

Compute derivatives with respect to $h_R$ $\rightarrow$ estimation of the critical exponent $\nu$.

Obtain size-dependent effective exponents $\rightarrow$ control scaling corrections (make use of the quotient method).

Perform high statistics in both directions: $L \leq 192$ in $d = 3$ ($L \geq 60$ in $d = 4$), # disorder samples in the range $(10^{-50}) \times 10^6$.
Ingredients in our approach (III): Computational scheme

- Use fluctuation-dissipation formalism to:
  - Compute *connected* correlation functions $\Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y}$.
  - Compute *disconnected* correlations $G_{xy} = \langle S_x S_y \rangle$.

For either correlator $\to$ second-moment correlation length.

- Perform re-weighting extrapolation on $h_R$.
- Compute derivatives with respect to $h_R \to$ estimation of the critical exponent $\nu$.
- Obtain size-dependent effective exponents $\to$ control scaling corrections (make use of the quotient method).

Perform high statistics in both directions: $L \leq 192$ in $d = 3$ ($L \geq 60$ in $d = 4$), # disorder samples in the range $(10^{-50}) \times 10^6$. 
Ingredients in our approach (III): Computational scheme

- Use fluctuation-dissipation formalism to:
  - Compute *connected* correlation functions $\Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y}$.
  - Compute as well *disconnected* correlations $G_{xy} = \langle S_x S_y \rangle$.

For either correlator $\rightarrow$ second-moment correlation length.
Perform re-weighting extrapolation on $h_R$.
Compute derivatives with respect to $h_R$ $\rightarrow$ estimation of the critical exponent $\nu$.
Obtain size-dependent effective exponents $\rightarrow$ control scaling corrections (make use of the quotient method).
Perform high statistics in both directions: $L \leq 192$ in $d = 3$ ($L \geq 60$ in $d = 4$), # disorder samples in the range $(10^{-50}) \times 10^6$. 
Use fluctuation-dissipation formalism to:

- Compute *connected* correlation functions $\Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y}$.
- Compute as well *disconnected* correlations $G_{xy} = \langle S_x S_y \rangle$.
- For *either* correlator $\rightarrow$ second-moment correlation length.
Ingredients in our approach (III): Computational scheme

- Use fluctuation-dissipation formalism to:
  - Compute *connected* correlation functions \( \Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y} \).
  - Compute as well *disconnected* correlations \( G_{xy} = \langle S_x S_y \rangle \).
  - For *either* correlator \( \rightarrow \) second-moment correlation length.
  - Perform re-weighting extrapolation on \( h_R \).
Ingredients in our approach (III): Computational scheme

- Use fluctuation-dissipation formalism to:
  - Compute *connected* correlation functions \( \Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y} \).
  - Compute as well *disconnected* correlations \( G_{xy} = \langle S_x S_y \rangle \).
  - For *either* correlator \( \rightarrow \) second-moment correlation length.
  - Perform re-weighting extrapolation on \( h_R \).
  - Compute derivatives with respect to \( h_R \) \( \rightarrow \) estimation of the critical exponent \( \nu \).
Use fluctuation-dissipation formalism to:

- Compute *connected* correlation functions $\Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y}$.
- Compute as well *disconnected* correlations $G_{xy} = \langle S_x S_y \rangle$.

For *either* correlator $\rightarrow$ second-moment correlation length.

- Perform re-weighting extrapolation on $h_R$.
- Compute derivatives with respect to $h_R$ $\rightarrow$ estimation of the critical exponent $\nu$.

- Obtain size-dependent effective exponents $\rightarrow$ control scaling corrections (make use of the quotient method).
Ingredients in our approach (III): Computational scheme

- Use fluctuation-dissipation formalism to:
  - Compute *connected* correlation functions \( \Gamma_{xy} = \frac{\partial \langle S_x \rangle}{\partial h_y} \).
  - Compute as well *disconnected* correlations \( G_{xy} = \langle S_x S_y \rangle \).
  - For *either* correlator \( \rightarrow \) second-moment correlation length.
  - Perform re-weighting extrapolation on \( h_R \).
  - Compute derivatives with respect to \( h_R \rightarrow \) estimation of the critical exponent \( \nu \).

- Obtain size-dependent effective exponents \( \rightarrow \) control scaling corrections (make use of the quotient method).

- Perform high statistics in both directions: \( L \leq 192 \) in \( d = 3 \) \( (L \geq 60 \) in \( d = 4 \)), \# disorder samples in the range \( (10 - 50) \times 10^6 \).
The computational challenge

- Overall goal: obtain some $5 \times 10^7$ ground states.
- Even with very efficient min-cut/max-flow algorithms it is crucial to optimize application.
Overall goal: obtain some $5 \times 10^7$ ground states.

Even with very efficient min-cut/max-flow algorithms it is crucial to optimize application.

Homemade code 10 times faster than library implementations: 2 minutes per Ground State on modern CPU: 2 million hours of CPU.
The computational challenge

- Overall goal: obtain some $5 \times 10^7$ ground states.
- Even with very efficient min-cut/max-flow algorithms it is crucial to optimize application.
- Homemade code 10 times faster than library implementations: 2 minutes per Ground State on modern CPU: 2 million hours of CPU.
- Beyond capabilities of any local resource
Overall goal: obtain some $5 \times 10^7$ ground states.

Even with very efficient min-cut/max-flow algorithms it is crucial to optimize application.

Homemade code 10 times faster than library implementations: 2 minutes per Ground State on modern CPU: 2 million hours of CPU.

Beyond capabilities of any local resource RES!!.
The computational challenge

- Overall goal: obtain some $5 \times 10^7$ ground states.
- Even with very efficient min-cut/max-flow algorithms it is crucial to optimize application.
- Homemade code **10 times faster** than library implementations: 2 minutes per Ground State on modern CPU: **2 million hours** of CPU.
- Beyond capabilities of any local resource RES!!.
- Major logistic problems to be faced: disk storage, massive I/O flow, data base of results...
• Overall goal: obtain some $5 \times 10^7$ ground states.
• Even with very efficient min-cut/max-flow algorithms it is crucial to optimize application.
• Homemade code 10 times faster than library implementations: 2 minutes per Ground State on modern CPU: 2 million hours of CPU.
• Beyond capabilities of any local resource RES!!.
• Major logistic problems to be faced: disk storage, massive I/O flow, data base of results... The MareNostrum could handle it all.
Crossings of the universal ratio $\xi/L$ (from connected $\Gamma_{xy}$)

![Graph showing crossings of the universal ratio $\xi/L$ for different values of $L$. The graph includes data points for $L = 16, 24, 32, 48, 64, 96$. The x-axis represents $\sigma$ and the y-axis represents $\xi/L$. The lines for each $L$ value show the trend as $\sigma$ increases.](image-url)

Poissonian RFIM
Mind the very strong scaling corrections!

$\xi/L$ at the crossing points: different models differ at fixed $L$. 

![Graph showing $\xi/L$ vs $L^{-1}$]
Mind the very strong scaling corrections!

\( \xi/L \) at the crossing points: different models differ at fixed \( L \).

One needs extrapolation to large \( L \).
Universality in the $d = 3$ RFIM

$$A + BL^{-\omega} + CL^{-2\omega} + DL^{-3\omega}; \quad L_{\text{min}} = 24; \quad \omega = 0.52 \pm 0.11; \quad \chi^2/dof = 18.83/14, \quad Q = 0.17 \text{ (full covariance-matrix!)}$$
Universality in the $d = 3$ RFIM

$$A + BL^{-\omega} + CL^{-2\omega} + DL^{-3\omega}; \ L_{\text{min}} = 24; \ \omega = 0.52 \pm 0.11; \ \chi^2/dof = 18.83/14, \ Q = 0.17 \text{ (full covariance-matrix!)}$$
The same approach can be carried out in $d=4$
Extrapolation of $\nu$

\[ \nu_L = \nu + B L^{-\omega} ; \quad L_{\text{min}} = 32 ; \quad \omega = 0.52 ; \]
\[ \chi^2 / \text{dof} = 12.52 / 10, \quad Q = 0.25 \]
Final estimate: $\nu = 1.38 \pm 0.10$
Extrapolation of $\eta$

\[ \eta_L = \eta + BL^{-\omega} ; \ L_{\text{min}} = 32 ; \ \omega = 0.52 ; \ \chi^2/dof = 10/9, \ Q = 0.35 \]

Final estimate: $\eta = 0.5153 \pm 0.0009$
Extrapolation of $2\eta - \bar{\eta}$

$$(2\eta - \bar{\eta})|_L = BL^{-\omega} ; L_{\text{min}} = 16 ; \omega = 0.52 ; \chi^2/dof = 18.26/18, \, Q = 0.44$$
The phase diagram of the RFIM is seemingly ruled by a **single fixed point**:
The phase diagram of the RFIM is seemingly ruled by a single fixed point:

Existence of strong scaling corrections that need to be carefully monitored. Very accurate computation of anomalous dimensions $\eta$, $\tilde{\eta}$. 
Conclusions

- The phase diagram of the RFIM is seemingly ruled by a **single fixed point**:

  ![Phase Diagram](image)

- Existence of **strong scaling corrections** that need to be carefully monitored. Very accurate computation of anomalous dimensions $\eta, \bar{\eta}$.

- The **two-exponent scaling scenario holds** within an accuracy of two parts in a thousand ($2/1000$) in $d = 3$. Analysis for $d = 4$ on their way.