Speed-up Solving Linear Systems via Composition of Clans

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Speed-Up Solving Systems on 16 Nodes
Key Features

- Speed-up solving (especially Diophantine) systems of linear algebraic equations
- Sparse systems of specific form, namely “well decomposable into clans”
- Concept of a sign forms clans of equations
- Applicable to other algebraic structures with sign
Form of Obtained Matrix

\[ A = \begin{bmatrix}
  A^{0,1} & \hat{A}^1 & 0 & 0 & 0 & 0 \\
  A^{0,2} & 0 & \hat{A}^2 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  A^{0,k} & 0 & 0 & 0 & 0 & \hat{A}^k \\
\end{bmatrix} \]
Divide and Sway

• Decompose a given system into its clans
• Solve a system for each clan
• Solve a system of clans composition
• Or collapse the decomposition graph solving a system for each contracted edge
• Obtain a result in feasible time
A Clan – Transitive Closure of Nearness Relation

Two equations are *near* if they contain the same variable having nonzero coefficients of the same sign.
Decomposition into Clans

\[
\begin{align*}
-x_1 + x_2 - x_{18} &= 0 \\
-x_2 + x_3 - x_{15} + x_{18} &= 0 \\
-x_2 + x_4 - x_{14} + x_{18} &= 0 \\
-x_4 + x_5 + x_{14} - x_{18} &= 0 \\
-x_3 + x_5 + x_{15} - x_{18} &= 0 \\
-x_5 + x_6 - x_{16} + x_{18} &= 0 \\
-x_6 + x_7 + x_{16} - x_{19} &= 0 \\
x_1 - x_7 + x_{19} &= 0 \\
-x_8 + x_9 - x_{19} &= 0 \\
-x_9 + x_{10} - x_{17} + x_{19} &= 0 \\
-x_{10} + x_{11} + x_{17} - x_{18} &= 0 \\
x_{11} + x_{12} - x_{15} + x_{18} &= 0 \\
x_{12} + x_{13} + x_{15} - x_{18} &= 0 \\
x_{13} + x_8 + x_{18} &= 0 \\
\end{align*}
\]

\[
\begin{align*}
-x_2 + x_3 - x_{15} + x_{18} &= 0 \\
-x_2 + x_4 - x_{14} + x_{18} &= 0 \\
-x_5 + x_6 - x_{16} + x_{18} &= 0 \\
-x_{11} + x_{12} - x_{15} + x_{18} &= 0 \\
-x_{13} + x_8 + x_{18} &= 0 \\
\end{align*}
\]

\[
\begin{align*}
-x_1 + x_2 - x_{18} &= 0 \\
-x_4 + x_5 + x_{14} - x_{18} &= 0 \\
-x_3 + x_5 + x_{15} - x_{18} &= 0 \\
-x_{10} + x_{11} + x_{17} - x_{18} &= 0 \\
-x_{12} + x_{13} + x_{15} - x_{18} &= 0 \\
\end{align*}
\]

\[
\begin{align*}
x_1 - x_7 + x_{19} &= 0 \\
-x_9 + x_{10} - x_{17} + x_{19} &= 0 \\
-x_6 + x_7 + x_{16} - x_{19} &= 0 \\
-x_8 + x_9 - x_{19} &= 0 \\
\end{align*}
\]
Systems and Directed Bipartite Graphs

Equation – transition (rectangle)

Variable – place (circle)

Positive sign – incoming arc of a place

Negative sign – outgoing arc of a place
Decomposition Graph

\[
\begin{align*}
\text{C1:} & \quad \begin{cases} 
-x_2 + x_3 - x_{15} + x_{18} = 0 \\
-x_2 + x_4 - x_{14} + x_{18} = 0 \\
-x_5 + x_6 - x_{16} + x_{18} = 0 \\
-x_{11} + x_{12} - x_{15} + x_{18} = 0 \\
-x_{13} + x_8 + x_{18} = 0 
\end{cases} \\
\text{C2:} & \quad \begin{cases} 
-x_1 + x_2 - x_{18} = 0 \\
-x_4 + x_5 + x_{14} - x_{18} = 0 \\
-x_3 + x_5 + x_{15} - x_{18} = 0 \\
-x_{10} + x_{11} + x_{17} - x_{18} = 0 \\
-x_{12} + x_{13} + x_{15} - x_{18} = 0 
\end{cases} \\
\text{C3:} & \quad \begin{cases} 
x_1 - x_7 + x_{19} = 0 \\
-x_9 + x_{10} - x_{17} + x_{19} = 0 
\end{cases} \\
\text{C4:} & \quad \begin{cases} 
-x_6 + x_7 + x_{16} - x_{19} = 0 \\
-x_8 + x_9 - x_{19} = 0 
\end{cases}
\end{align*}
\]
Collapse of Decomposition Graph

I.

II.
Decomposition into Clans as Matrix Reordering

- Clan – subset of equations
- Decomposition into clans – reordering of rows
- Linear complexity in the number of nonnegative elements
- Classification of variables into contact and internal (on clans) reorders columns
- Combination of a block-column and a block diagonal matrices
Systems of Equations (Inequalities)

\[ A \cdot \bar{x} = \bar{b} \]

its general solution

\[ \bar{x} = \bar{x}' + G \cdot \bar{y} \]

Consider a system as a predicate

\[ S(\bar{x}) = L_1(\bar{x}) \land L_2(\bar{x}) \land \ldots \land L_m(\bar{x}) \]

\[ L_i(\bar{x}) = (\bar{a}^i \cdot \bar{x} = 0), \quad \mathcal{I} = \{L_i\} \]
Relations on the Set of Equations

Relation of nearness: \( L_i \circ L_j, \)
\[ \exists x_k \in X : \ a_{i,k}, a_{j,k} \neq 0, \ \text{sign}(a_{i,k}) = \text{sign}(a_{j,k}) \]

**Statement.** The relation of nearness is reflexive and symmetric.

Relation of clan: \( L_i \circ L_j \)
\[ L_{l_1}, L_{l_2}, ..., L_{l_k} : \ L_i \circ L_{l_1} \circ ... \circ L_{l_k} \circ L_j \]

**Theorem.** The relation of clan is an equivalence relation (reflexive, symmetric, and transitive).

**Corollary.** Relation of clan defines a *partition* of the set of equations; an element of this partition is named a *clan.*
Classification of Variables

Variables of a clan: \( X^j \)

\[ X^j = X(C^j) = \{ x_i \mid x_i \in X, \exists L_k \in C^j : a_{k,i} \neq 0 \} \]

Internal variables of a clan: \( \tilde{X}^j \)

\[ x_i \in X(C^j), \quad \forall C^l, l \neq j : \quad x_i \notin X^l \]

Contact variables: \( X^0 \)

\[ \exists C^j, C^l : \quad x_i \in X^j, \quad x_i \in X^l \]

Contact variables of a clan: \( \tilde{X}^j \)

\[ X^j = \tilde{X}^j \cup \tilde{X}^j, \quad \tilde{X}^j \cap \tilde{X}^j = \emptyset \]

**Theorem.** A contact variable belongs to two clans exactly entering one clan with sign plus and the other clan with sign minus.
Decomposition of System Matrix

<table>
<thead>
<tr>
<th>Clans/variables</th>
<th>$X^0$</th>
<th>$\hat{X}^1$</th>
<th>$\hat{X}^2$</th>
<th>$\ldots$</th>
<th>$\hat{X}^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^1$</td>
<td>$A^{0,1}$</td>
<td>$\hat{A}^1$</td>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>$C^2$</td>
<td>$A^{0,2}$</td>
<td>0</td>
<td>$\hat{A}^2$</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>$C^k$</td>
<td>$A^{0,k}$</td>
<td>0</td>
<td>0</td>
<td>$\ldots$</td>
<td>$\hat{A}^k$</td>
</tr>
</tbody>
</table>
Composition of Clans

1. Solve the system separately for each clan: \( \bar{x}^j = G^j \cdot \bar{y}^j \)

\[
A^j \cdot \bar{x}^j = 0, \quad A^j = \begin{bmatrix} \bar{A}^j \\ \bar{A}^j \end{bmatrix}, \quad \bar{x}^j = \begin{bmatrix} \bar{x}^j \\ \bar{x}^j \end{bmatrix}
\]

2. Solve a system of composition of clans for contact variables:

\( G_i^j \cdot \bar{y}^j = G_i^l \cdot \bar{y}^l \) or \( F \cdot \bar{y} = 0 \): \( \bar{y} = R \cdot \bar{z} \)

3. Recover sought solutions:

\[
\bar{x} = G \cdot \bar{y}, \quad G = \begin{bmatrix} J^1 & \hat{G}^1 & 0 & 0 & 0 \\ J^2 & 0 & \hat{G}^2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ J^k & 0 & 0 & 0 & \hat{G}^k \end{bmatrix}^T, \quad \bar{x} = G \cdot R \cdot \bar{z},
\]
General Solutions Obtained via Composition of Clans

**Theorem 1.** A general solution of homogeneous system is:

\[ \bar{x} = H \cdot \bar{z}, \quad H = G \cdot R \]

**Theorem 2.** A general solution of heterogeneous system is:

\[ \bar{x} = \bar{y}'' + H \cdot \bar{z}, \quad \bar{y}'' = \bar{x}' + G \cdot \bar{y}', \quad H = G \cdot R \]

**Statement.** Speed-up of computations is about:

\[ \frac{M(q)}{k \cdot nz + k \cdot M(p)} \]

For exponential methods – exponential speed-up:

\[ O(2^{q-p}) \]
Example: Decomposition into Clans

\[
A = \begin{pmatrix}
1 & 0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 2 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
C^1 = \{L_1, L_2, L_5, L_6\} \quad C^2 = \{L_3, L_4, L_7, L_8, L_9\}
\]

\[
X^1 = \{x_3, x_6, x_8, x_{10}, x_1, x_2, x_7\} \quad X^2 = \{x_3, x_6, x_8, x_{10}, x_4, x_5, x_9\}
\]

\[
\tilde{X}^1 = \{x_1, x_2, x_7\} \quad \tilde{X}^2 = \{x_4, x_5, x_9\}
\]

\[
X^0 = \tilde{X}^1 = \tilde{X}^2 = \{x_3, x_6, x_8, x_{10}\}
\]
Example: Renumeration of Equations and Variables

\[ nx = \begin{pmatrix} 3 & 6 & 8 & 10 & 1 & 2 & 7 & 4 & 5 & 9 \end{pmatrix} \]

\[ nL = \begin{pmatrix} 1 & 2 & 5 & 6 & 3 & 4 & 7 & 8 & 9 \end{pmatrix} \]

\[ A = \begin{pmatrix} 2 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix} \]
Example: Solution of Systems for Clans

\[ G^1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^T, \]

\[ \bar{y}^1 = (y_1^1, y_2^1, y_3^1)^T \]

\[ G^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^T, \]

\[ \bar{y}^2 = (y_1^2, y_2^2)^T \]
Example: Solution of System for Contact Variables

\[
\begin{align*}
  y_1^1 - y_1^2 &= 0, \\
  y_1^1 - y_1^2 &= 0, \\
  y_2^1 - y_1^2 &= 0, \\
  y_2^1 - y_1^2 &= 0.
\end{align*}
\]

\[
R = \begin{bmatrix}
  1 & 1 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1
\end{bmatrix}^T
\]
Example: Composition of Source System Solution

\[ G = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix} \]

or \[ G = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix} \]

\[ H = R \cdot G = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix} \]
Sequential Contraction of Graphs as a Scheme of Solving System

Graph of system decomposition into its clans: \( G = (V, E, W) \)

\[ V = \{v\}, \quad v \leftrightarrow C \quad \text{vertices correspond to clans} \]

\[ E \subseteq V \times V \quad \text{edges connect clans having common contact variables} \]

\[ v_1 v_2 \in E \iff \exists x \in X^0 : (I(x) = C^1 \land O(x) = C^2) \lor (I(x) = C^2 \land O(x) = C^1) \]

\[ W : (V \rightarrow \mathbb{N}) \cup (E \rightarrow \mathbb{N}) \quad \text{weight function;} \]

\[ w(v) \quad \text{number of clan variables;} \quad w(v) \geq \sum_w w(v, u) \]

\[ w(v, u) \quad \text{number of contact variables;} \]

Collapse of graph: \( G = G^0 \xrightarrow{\text{Collapse}} G^1 \xrightarrow{\text{Collapse}} G^2 \ldots \xrightarrow{\text{Collapse}} G^k \)
Collapse of Subgraphs
Edge collapse of graph

Collapse width 15 – dimension of systems.
An Exhaustive Search of Edge Collapse
A Partial Lattice of Collapse

\[ e_1^i \ll e_3^{i+1} \iff e_3^{i+1} = e_1^i \lor e_3^{i+1} = e_1^i + e_2^i \]

\[ p_i = p_{i-1} - 1 - t \]  - number of edges

\[ t \]  - number of triangles

**Statement.** Each edge on a step of a collapse is a sum of some edges of the source graph.
Comparing Heuristic Strategies of Edge Collapse
(maximal, random, and minimal edge)
## Comparison of Collapse Strategy for Random Graphs

<table>
<thead>
<tr>
<th>Number of graph vertices</th>
<th>Density of graph (%)</th>
<th>Width of simultaneous collapse</th>
<th>Width of sequential collapse</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Maximal edge Width</td>
<td>%</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>442</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>869</td>
<td>66</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>1372</td>
<td>102</td>
</tr>
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<td></td>
<td>80</td>
<td>1825</td>
<td>160</td>
</tr>
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<td>40</td>
<td>20</td>
<td>1836</td>
<td>73</td>
</tr>
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</tr>
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<td></td>
<td>80</td>
<td>7354</td>
<td>314</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>11602</td>
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<td></td>
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<td>200</td>
<td>20</td>
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</tr>
<tr>
<td></td>
<td>80</td>
<td>183652</td>
<td>1486</td>
</tr>
</tbody>
</table>
Software

• *Deborah* – decomposition into clans, 2004
• *Adriana* – solving a homogenous system via (a) simultaneous or (b) sequential composition of clans, 2005
• *ParAd* – solving a homogenous system via (a) simultaneous or (b) parallel-sequential composition of clans on *parallel architectures*, 2017
Composition of Clans on Parallel Architectures

- Decompose a system into clans
- Solve a system for clan 1
- Solve a system for clan k
- Solve a system of clans composition

MPI

OpenMP
Parallel-sequential Composition of Clans
ParAd – Parallel Adriana

ParAd

Decomposition into clans

Simultaneous composition of clans

ParTou

hnf_solve

Parallel-sequential composition of clans

zsolve

Solvers of linear systems
Protocols of Master-Worker Communication

(a) Sending system

Worker

MPI_Send

MPI_Recv

MPI_Recv

MPI_Recv

Worker

'M' 51

'W' 52

M

lenm 53

bufm 54

MPI_Send

MPI_Send

MPI_Send

MPI_Send

Master

MPI_Recv

MPI_Send

MPI_Send

MPI_Send

(b) Receiving solution

Master

MPI_Recv

MPI_Send

MPI_Send

MPI_Send

lenr 57

bufr 58
Master-Worker Basic Communication Model
Parallel-Sequential Composition Communication Model
Run ParAd

• Run with mpirun

```bash
>mpirun -n 5 ./ParAd -c -r zsolve tcp.spm tcp-pi.spm
>mpirun -n 10 ./ParAd -s -t -d 1 tcp.spm tcp-ti.spm
```

• Run with Slurm

```bash
>srun -N 10 ./ParAd -s -t -d 1 tcp.spm tcp-ti.spm
```

• SPM – simple sparse matrix format:

```
i j a[i][j]
```

• Check decomposability (Matrix Market Format)

```bash
>toclans lp_cre_d.mtx
```
Aggregation of Clans for Workload Balancing

• A clan is a sum of minimal clans
• The maximal clan size restricts granulation
• Many small clans lead to heavy communication load
• Balancing: create clans having size close to the maximal
• Key: -a val
• Aggregation steeds-up about 20%
Solving Systems over Real Numbers

• How to solve a linear system for a non-square matrix (what software to use)\
• A variant: LAPACK, SVD
• A problem – accumulation of errors
• Preference to simultaneous composition
• Clans with SVD speed-up about 2 times on 16 nodes
Conclusions

- Composition of clans speeds-up solving linear systems of equations
- Decomposition into clans is linear in the number of nonnegative elements
- Technique is applicable to sparse-matrices decomposable into clans
- Many application area matrices are decomposable into clans
Basic references


• A library that implements composition of clans independently from data types and solvers
• Multi-core implementation of decomposition
• Multi-core implementation of sparse matrix multiplication
• Solve heterogeneous systems and inequalities

http://member.acm.org/~daze