F046

Construction and Analysis of an Optimized Compact Finite Difference Scheme for RTM

H. Zhou* (Repsol USA), Q. Liao (Repsol USA) & F. Ortigosa (Repsol USA)

SUMMARY

We present an optimized compact finite-difference scheme that has a fourth-order of accuracy but with higher resolving power over a larger range of wavenumbers compared to other traditional finite-difference schemes. In order to realize that, we borrow ideas from Lele s compact scheme (Lele, 1992) and Tam s DRP schemes (Tam and Webb, 1993). I.e., we design an optimized finite-difference scheme that uses short stencils but with an optimized coefficient. This way it may help us take advantage of the computer caches without losing higher resolving power. A detailed Fourier analysis on the proposed scheme has been analyzed. A migration impulse response has been tested using our optimized compact finite-difference scheme.

Introduction

3D pre-stack depth migration techniques have been widely used for subsurface imaging. Reverse-time migration (RTM) method utilizes two-way scalar wave equation (Baysal et al., 1983; McMechan, 1983). As such it does not suffer from the dip limitations inherent in one-way wave equation techniques, thus enabling imaging of overturned reflections and other complex structures. However, in order to get less dispersion images, RTM requires higher-order finite difference scheme that generally consists of long stencils. This makes RTM notoriously known to be impractical due to its high compute intensive.

One of the most basic problems in finite-difference analysis is that not all the scales supported by a grid are well resolved by a finite-difference scheme. In numerical solutions of scalar wave equation it may happen that the velocity of a wave depends on the frequency. If the grid is too coarse, the signals computed by the finite-difference method become strongly dispersed. This effect, known as “grid dispersion”, must be taken into account in order to avoid erroneous interpretation of seismograms obtained by FD techniques.

Finite-difference works by approximating the derivatives of a function at a point by using the function values at some neighbor points. The standard way to find the coefficients is to use Taylor’s Theorem. However, in the past few years, it has been recognized that difference approximation should represent exact result over a large range of length scales that can be realized on a given mesh, rather than their formal accuracy (i.e., truncation errors) (Vichnevetsky and Bowles, 1982).

In this abstract, we present an optimized compact finite-difference scheme that has a fourth-order of accuracy but with higher resolving power over a larger range of wavenumbers compared to other traditional finite-difference schemes. In order to realize that, we borrow ideas from Lele’s compact scheme (Lele, 1992) and Tam’s DRP schemes (Tam and Webb, 1993). I.e., we design an optimized finite-difference scheme that uses short stencils but with an optimized coefficient. This way it may help us take advantage of the computer caches without losing higher resolving power. A detailed Fourier analysis on the proposed scheme has been analyzed. A migration impulse response has been tested using our optimized compact finite-difference scheme.

Finite difference approximation of second spatial derivative

Scalar wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\ , \tag{1}$$

where $c(x, y, z)$ is wave propagation velocity and $u(x, y, z, t)$ is the wavefields. It provides the well-known dispersion relation for wave equation (1)

$$c^2 k^2 = \omega^2\ , \tag{2}$$

here, $k^2 = k_x^2 + k_y^2 + k_z^2$. And $\omega = 2\pi f$ is the angular frequency with f is the frequency. The phase velocity is the velocity a monochromatic wave with wave number k propagates with and is given by

$$c_p = \frac{\omega}{|k|}$$

For scalar wave equation, we have $c_p = c$, which is dispersion-free.

Without loss of generality, in the following, we will only discuss second derivative

$u''(x) = \partial^2 u / \partial x^2$ in one dimension. Derivations for other space dimensions can be similarly obtained. Similarly, we will not consider the effect of discretizations in time direction.
Let u_i'' represents the finite difference approximation to the second derivative $u''(x)$ at grid location x_i .According to Lele (1992), a tri-diagonal approximation u_i'' of the second derivative is given by

$$\alpha u_{i-1}'' + u_i'' + \alpha u_{i+1}'' = a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2} , \qquad (3)$$

Here $a, b,$ and α are finite-difference coefficients that needs to be determined. And h is the grid space interval. When α is non-zero, this is an implicit formulations based on Pade-type development. A family of finite-difference schemes can be generated that can maintain fourth-order accuracy. In order to do that, the relations between the coefficients $a, b,$ and α are derived by matching the Taylor series coefficients until fourth order accuracy. For equation (3), we get the following relations:

$$a = \tfrac{4}{3}(1-\alpha),$$
$$b = \tfrac{1}{3}(-1+10\alpha)$$

Different choices of these coefficients determine the formal accuracy and the resolving power of the finite-difference schemes. For example, some special cases of equation (3) are listed below:

a) $\alpha = 0$, this becomes standard fourth-order central finite difference scheme
b) $\alpha = 1/10$, we have $b = 0$. This is the classical Pade scheme that has been extensively used in finite-difference for one-way wave equation (Claerbout, 1985).
c) $\alpha = 2/11$, we have $a = 1/11$, and $b = 3/11$, This corresponds to a sixth-order tri-diagonal finite-difference scheme.

Accuracy of equation (3) can be measured directly from the spatial derivative by a Fourier analysis. Let $w = kh$ be the modified or scaled wavenumber. Obviously, from Fourier analysis, w^2 is the true wavenumber for the second derivative. Similarly, the numerical approximation $w''(w)$ of second derivative can be found by Fourier analysis of equation (3)

$$w''(w) = \frac{2a[1-\cos(w)] + b/2[1-\cos(2w)]}{1 + 2\alpha\cos(w)} \qquad (4)$$

Here, $w''(w)$ is called effective wavenumber for the second derivative and will be used for the analysis of finite difference errors. The difference between $w''(w)$ and w^2 is a measure of error in the second derivative approximation. To ensure that the Fourier transform of the finite difference scheme is a good approximation of the partial derivative, the effective wavenumber $w''(w)$ should coincide with the corresponding true wavenumber w^2 over as wide a range of wavenumber (i.e., $0 \le w < \pi$) as possible. Minimizing the error on the effective wavenumber is equivalent to control the phase-velocity error that is relevant for a simple harmonic plane wave. Therefore, optimized schemes can be obtained by minimizing the error on the effective wavenumber $w''(w)$, which are defined as the integral error

$$\min_{\alpha} E(\alpha) = \int_{w_l}^{w_u} \left| w'' - w^2 \right|^2 \xi(w)dw$$

over a large wavenumber range $w_l = k_l h \le w = kh \le w_u = k_u h$, or also by minimizing the relative error

$$\min_{\alpha} E(\alpha) = \int_{w_l}^{w_u} \left| \frac{w'' - w^2}{w^2} \right| \eta(w)dw$$

Here, $\xi(w)$ and $\eta(w)$ are the weight functions for the integrals, which can be used to boost or reduce certain range of wavenumbers. It is necessary to find the optimum values of the coefficients that minimize the errors. Plots of the effective wavenumber $w''(w)$ against the modified or scaled wavenumber w are represented in Figure 1 for a variety of schemes. It is evident that compared to other standard central differences our optimized compact scheme stay close to the exact differentiation over a wide range of wavenumbers. The coefficients for optimized finite-difference scheme are $a = 1.06195$, $b = 0.345126$, $c = 0.203538$.

The effectiveness of a finite-difference scheme can also be shown by the Points-Per-Wavelength G. Assume wavelength $\lambda = \frac{2\pi}{k} = \frac{c}{f}$.Therefore, G can be represented as $G = \frac{\lambda}{h} = \frac{2\pi}{kh}$ [Alford et al. (1974)], where h is the grid interval. From equation (4), we obtain

$$w''(G) = \frac{2a[1-\cos(2\pi/G)] + b/2[1-\cos(4\pi/G)]}{1+2\alpha\cos(2\pi/G)} \qquad (5)$$

This represents the minimum grid points that the algorithm needs so that the waves propagate with less numerical dispersions. Figure 2 shows the relative errors of the effective wavenumber with respect to G for different finite-difference schemes. Obviously, our proposed scheme has fewer Points-Per-Wavelength G.

Example

Figure 3 shows a vertical slice of the impulse response for 3D depth migration using our fourth-order optimized compact finite-difference scheme. The source is Ricker wavelet with peak frequency is 20Hz. The source is put right in the middle of the surface and two receivers are put at two sides with equal distance from source location. Note the clean migration image computed with the proposed finite-difference scheme.

Discussions and conclusions

An optimized fourth-order compact finite difference scheme for the evaluation of second derivative has been presented and analyzed. Comparisons were made throughout with other well-known schemes. In stead of emphasizing on accurate resolution of a single monochromatic wave, we focus on improving the representation of a range of wavenumbers. Fourier analysis proves the high resolving resolution for this kind of optimization. Because of the application of shorter stencils for approximations of second derivative of scalar wave equation, the data for finite-difference computations has the great potential of fitting into computer caches and at the same time improving resolving power of the scheme. However, it doesn't mean that this optimized scheme is perfect. One of the problems is the LU-type of tri-diagonal solvers. Because of this it may be not easy for the scheme to take full advantage of vectorization mechanism compared to traditional explicit finite-difference schemes. Further tests need to be done to validate the efficiency of the proposed optimized compact scheme.

Acknowledgment

The first author (H.Z.) would like to thank Kaichuan Chen in GX Technology for kindly provide the author Lele's and Tam's papers.

References

Alford, R.M., Kelly, K.R., and Boore, D.M., [1974] Accuracy of finite-difference modeling of the acoustic wave equation, Geophysics, 39, pp. 834-842.

Baysal, E. Kosloff, D.D. and Sherwood, J.W.C., [1983] Reverse time migration: Geophysics, **48**, 1514-1524.

Claerbout, J. F. [1985] Imaging the earth’s interior, Blackwell Science Inc..

Lele, S. [1992] Compact finite difference schemes with spectral-like resolution, Journal of computational physics, 103, pp. 16-42.

McMechan, G.A., [1983] Migration by extrapolation of time-dependent boundary values: Geophysical Prospecting, **31**, 413-420.

Tam, C.K.W. and Webb, J.C., [1993] Dispersion-relation-preserving schemes for computational aeroacoustic, Journal of computational physics, 107, pp. 262-281.

Vichnevetsky, R. and Bowles, J.B., [1982] Fourier analysis of numerical approximations of hyperbolic equations, SIAM Philadelphia.

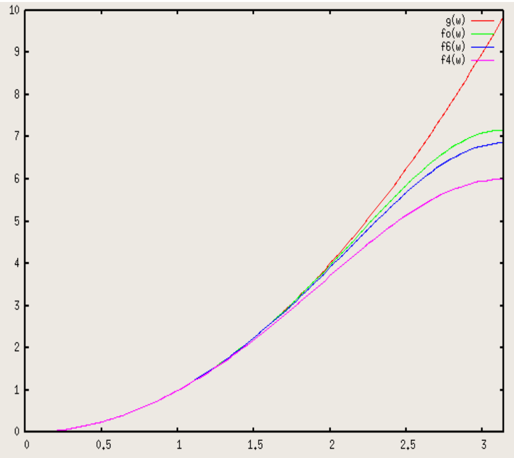

Fig. 1. Relation of $w''(w)$ and w^2 .

Red color is for accurate result,
Green is for our optimized scheme,
Blue is for 6-th order scheme, and
Pink is for 4-th order scheme.

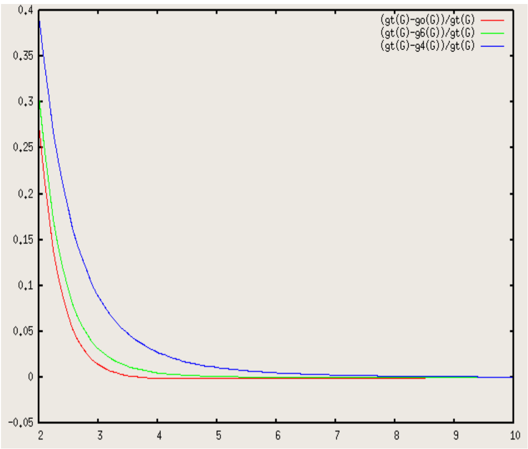

Fig. 2,Relation between $(w''(w)-w^2)/w^2$ and G .

Blue is for 4th order scheme.
Green is for 6th order scheme
Red is for our scheme.

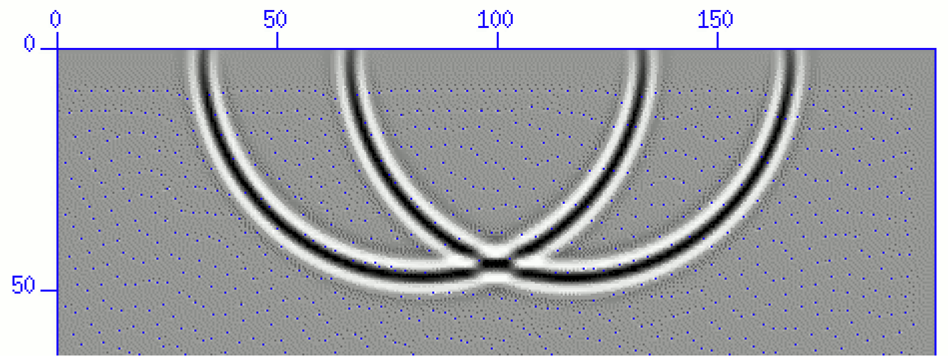

Fig. 3, vertical slice of the impulse response for 3D prestack depth migration.